Storage States in Ultracold Collective Atoms

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Abstract

We present a complete description of atomic storage states which may appear in the electromagnetically induced transparency (EIT). We include the spatial coherence in the atomic collective operators and the atomic storage states, and show the atomic storage states are Dicke states with the maximum cooperation number. In some limits, a set of multimode atomic storage states has been established in correspondence with multimode Fock states of electromagnetic field. This gives better understanding of that, in EIT, the optical coherent information can be preserved and recovered.

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1 introduction

In the interaction between optical field and atoms, a photon can be absorbed by atom and, then, the excited atom re-emits a photon by spontaneous or stimulated emissions. In this process the atom stores the energy of the field and releases it back to the field. Recently, the theoretical and experimental studies have shown that both quantum state and coherent information of field can be stored in atomic medium [1]-[6]. A very recent experiment witnesses

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that a signal pulse of light is stored in the ultracold collective atoms for up to a millisecond [1].

The basic scheme for storage of light information is carried out by the electromagnetically induced transparency (EIT) [7],[8]. N three-level atoms with one upper level $|a\rangle$ and two lower levels $|b\rangle$ and $|c\rangle$ interact resonantly with both the signal and the control beams. The weak signal beam and the strong control beam drive the atomic transitions $|a\rangle - |b\rangle$ and $|a\rangle - |c\rangle$, respectively. The early investigations have shown that EIT permits the propagation of the signal light through an otherwise opaque atomic medium and the light group velocity of the signal pulse is greatly reduced [9]-[11]. In the recent experiment [1], when the control beam turns off, the signal pulse is stopped and stored in atomic medium. This effect can be well understood by the dark states which are the eigenstates of EIT interaction Hamiltonian. The dark states are combination states of the photon state and the atomic storage state. The bosonic quasiparticles in the dark state are called polaritons. When the strength of the control field is changed adiabatically, both quantum state and coherent information transfer between the signal field and the collective atoms [3],[4]. A very recent theoretical study has shown that adiabatically changing of the control field is not necessary, and even a fast switching of the control field can be used in storage and retrieval of quantum information. [12]

In the early theoretical work,[13] Dicke has studied firstly coherence and cooperation effects in atomic ensemble. He defined collective atomic operators as sum of all individual atomic operators, retaining the properties of angular momentum. The Dicke states are the eigenstates of the angular momentum operators. In Dicke's paper, he considered two cases: the gas volumes have dimensions smaller and larger than the radiation wavelength. For the latter case, the spatial coherence has been included into collective atomic operators.

In this paper, we present a complete description of atomic storage state. Similarly as the second case of Dicke's work, we include spatial coherence into the collective atomic lower and upper operators. But we emphasize that, in some limits, they behave as multimode bosonic operators. Aside from the way of the original definition of Dicke states, we introduce atomic storage states with explicit expressions by containing spatial coherence of radiation field, and indicate they are Dicke states with the maximum cooperation number. The significant fact is that, under the low excitation limit, a set of multimode atomic storage states is established in correspondence with multimode Fock states of electromagnetic field. This gives a better understanding of that in EIT interaction both quantum and coherent information of electromagnetic field can be preserved in atomic medium.

2 atomic collective operators with the bosonic commutation

We consider N ultracold collective atoms which are approximately still at their positions. At very low temperature close to the critical temperature for Bose-Einstein condensation,[1] the average kinetic energy of atoms is greatly reduced. On the other hand, at low temperature, atoms are dense within a wavelength of optical field. The free path of atom is much less than the wavelength, it hence confines the scope of atomic motion. What the "still atoms" mean that, in the characteristic time of the system, the scale of motion for the centre-of-mass of atoms is much less than the wavelength of the optical field involved. The two levels of atoms $|b\rangle$ and $|c\rangle$ interact with some optical field of wavevector k. We assume that N is a large number and the most population of the atoms are in the level $|b\rangle$ throughout system evolution, so that the completeness relation is

$$N = \sum_{j=1}^{N} (|b_j\rangle\langle b_j| + |c_j\rangle\langle c_j|) \simeq \sum_{j=1}^{N} |b_j\rangle\langle b_j|. \tag{1}$$

It is not necessary that $|b\rangle$ is the ground state, for instance, in the case of EIT the level $|c\rangle$ can be lower or equal to $|b\rangle$. For the sake of convenience, we call $|b\rangle$ the "ground" state and $|c\rangle$ the "excited" state.

In the interaction between field and atomic medium, the spatial coherence of the field affects only the local atoms. In the approximation of "still atoms", the j-th atom located at position z_j suffers a local field strength with a phase $\exp(ikz_j)$. For this reason, we define the lower and the upper operators of the collective atoms as

$$\begin{aligned} \sigma_k &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |b_j\rangle\langle c_j| \exp(-ikz_j), \\ \sigma_k^\dagger &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |c_j\rangle\langle b_j| \exp(ikz_j), \end{aligned} \tag{2}$$

where k is the wavevector of the optical field interacting with the transition $|b\rangle - |c\rangle$. We notice that this kind of collective atomic operators containing spatial coherence was firstly introduced by Dicke.[13] In the EIT case, a Raman transition occurs between $|b\rangle$ and $|c\rangle$, k should be replaced by $k_s - k_c$ where k_s and k_c are respectively the wavevectors of the signal field and the control field. It may also be applied to a two-photon cascade transition so long as k is replaced by $k_1 + k_2$.

The commutation relations for these atomic operators are written as

$$\begin{aligned} [\sigma_k, \sigma_{k'}] &= [\sigma_k^\dagger, \sigma_{k'}^\dagger] = 0, && (3) \\ [\sigma_k, \sigma_{k'}^\dagger] &= (1/N)\sum_{j=1}^{N}(|b_j\rangle\langle b_j| - |c_j\rangle\langle c_j|)\exp[-i(k-k')z_j]. && (4) \end{aligned}$$

The exact commutation of Eq. (4) for the same mode is readily obtained

$$[\sigma_k, \sigma_k^\dagger] = (1/N)\sum_{j=1}^{N}(|b_j\rangle\langle b_j| - |c_j\rangle\langle c_j|). \qquad (5)$$

If N is a very large number and the most atoms are populated in the level $|b\rangle$ throughout evolution, by applying Eq. (1), Eq. (4) is approximately reduced to

$$[\sigma_k, \sigma_{k'}^\dagger] \simeq (1/N)\sum_{j=1}^{N}\exp[-i(k-k')z_j]. \qquad (6)$$

Assume that the atoms are in a string and the average interval of the adjacent atoms is d, which is much less than the optical wavelength, i.e. $kd << 1$, one obtains

$$\sum_{j=1}^{N}\exp[ikz_j] = \sum_{j=1}^{N}\exp[ik(j-1)d] = \frac{1-\exp[ikNd]}{1-\exp[ikd]} \approx N\frac{\exp[ikL]-1}{ikL}, \qquad (7)$$

where $L = Nd$ is the length of the atomic medium. However, this result is also true for a volume of atomic gas which is considered as a continuous medium

$$\sum_{j=1}^{N}\exp[ikz_j] = \int_0^L \frac{N}{L}\exp[ikz]dz = N\frac{\exp[ikL]-1}{ikL}. \qquad (8)$$

In the case of that the length of the atomic medium is much larger than the optical wavelength, we obtain

$$\frac{1}{N}\sum_{j=1}^{N}\exp[ikz_j] = \begin{cases} 1 & (k=0), \\ 0 & (kL >> 1). \end{cases} \qquad (9)$$

By applying the above result to Eq. (6), one obtains the bosonic commutation relation for the collective atomic operators

$$[\sigma_k, \sigma_{k'}^\dagger] \simeq \delta_{kk'}, \tag{10}$$

where we should assume $(k-k')L >> 1$, or, equivalently, $\lambda-\lambda' >> \lambda^2/(2\pi L)$. For the parameters used in the experiment [1], $L = 339\mu m$ and $\lambda = 589.6nm$, so that $\lambda^2/(2\pi L) \approx 0.163nm$, Eq. (10) is a good approximation for distinguishable modes. We will see in the next section that the atomic collective operators behave the same as the creation and annihilation operators of electromagnetic field.

3 single-mode atomic storage states

The state that all the atoms are in the ground level is as a "vacuum" state, symbolized in Ref. [3] as

$$|C^0\rangle \equiv |b_1 b_2 \cdots b_N\rangle. \tag{11}$$

When the single-mode creation operators of the collective atoms apply to the "vacuum"state, one obtains

$$\begin{aligned}
(\sigma_k^\dagger)^n |C^0\rangle &= \frac{1}{\sqrt{N^n}} \left(\sum_{j=1}^{N} |c_j\rangle\langle b_j| \exp(ikz_j) \right)^n |b_1 b_2 \cdots b_N\rangle \qquad (12)\\
&= \frac{1}{\sqrt{N^n}} \sum_{\{i_n\}}{}' |c_{i_1} \cdots c_{i_n}\rangle\langle b_{i_1} \cdots b_{i_n}|b_1 b_2 \cdots b_N\rangle \exp[ik(z_{i_1} + \cdots + z_{i_n})] \\
&= \frac{1}{\sqrt{N^n}} \sum_{\{i_n\}}{}' |b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle \exp[ik(z_{i_1} + \cdots + z_{i_n})]
\end{aligned}$$

where $\sum'_{\{i_n\}}$ designates that, in the summation, any two of indices cannot be equal, because of $(|c_j\rangle\langle b_j|)^2|b_j\rangle = 0$. We note that some states in the summation of Eq. (12), for which the sequence in the index set $\{i_n\}$ is exchanged, are same and should put together. For example, $(i_1 = 1, i_2 = 2, i_3, \cdots, i_n)$ and $(i_1 = 2, i_2 = 1, i_3, \cdots, i_n)$ display the same state. For an ensemble $\{i_n\}$ of n elements, there are $n!$ permutations which form the same state. By eliminating these repeated terms in the summation, Eq. (12) can be replaced by

$$(\sigma_k^\dagger)^n |C^0\rangle = \frac{n!}{\sqrt{N^n}} \sum_{\{i_n\}}{}'' |b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle \exp[ik(z_{i_1} + \cdots + z_{i_n})], \tag{13}$$

where $\sum''_{\{i_n\}}$ is designated as

$$\sum_{\{i_n\}}{}'' \equiv \underbrace{\sum_{i_1=1}^{N-n+1}\sum_{i_2=2}^{N-n+2}\cdots\sum_{i_{n-1}=n-1}^{N-1}\sum_{i_n=n}^{N}}_{\{i_1<i_2<\cdots<i_{n-1}<i_n\}}. \tag{14}$$

In the summation of Eq. (13), it includes $\binom{N}{n} = N(N-1)\cdots(N-n+1)/n!$ terms. Now, we define a normalized atomic storage state

$$\begin{aligned}|C_k^n\rangle \;\; &= \;\; \sqrt{\frac{n!}{N(N-1)\cdots(N-n+1)}} \\ &\quad \times \sum_{\{i_n\}}{}''|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \exp[ik(z_{i_1}+\cdots+z_{i_n})].\end{aligned} \tag{15}$$

Obviously, the atomic storage states with different number of excitations are orthogonal each other

$$\langle C_k^n|C_k^m\rangle = \delta_{nm}. \tag{16}$$

In this definition, the superposition state of N collective atoms includes any possible combination of n atoms being in the level $|c\rangle$, while the corresponding spatial coherence is recorded in the phase of the wavefunction. Physically, it means that n photons can be stored by any combination of excited n atoms with an equal possibility, in correspondence with the nonlocality for photons. However, the coherent information of the field has been contained in the probability amplitudes. Note that, the atomic storage state with a definite wavevector k is independent of position z which disappears in the summation.

By using definition (15), Eq. (13) becomes

$$(\sigma_k^\dagger)^n|C^0\rangle = \sqrt{\frac{N(N-1)\cdots(N-n+1)}{N^n}}\sqrt{n!}|C_k^n\rangle. \tag{17}$$

It is easy to check that

$$\sigma_k^\dagger|C_k^n\rangle = \sqrt{1-\frac{n}{N}}\sqrt{n+1}|C_k^{n+1}\rangle. \tag{18}$$

The above two equations are exact. However, in the limit $N >> n$, their approximate expressions are

$$(\sigma_k^\dagger)^n|C^0\rangle \simeq \sqrt{n!}|C_k^n\rangle, \tag{19}$$

and

$$\sigma_k^\dagger|C_k^n\rangle \simeq \sqrt{n+1}|C_k^{n+1}\rangle. \tag{20}$$

The annihilation operator is applied to the ”vacuum”state

$$\sigma_k|C^0\rangle = 0. \tag{21}$$

In the appendix A, the general formula for the annihilation operator are proved as

$$\sigma_k|C_k^n\rangle = \sqrt{1-\frac{n-1}{N}}\sqrt{n}|C_k^{n-1}\rangle \simeq \sqrt{n}|C_k^{n-1}\rangle. \tag{22}$$

Equations (18) and (22) give immediately

$$\sigma_k^\dagger\sigma_k|C_k^n\rangle = (1-\frac{n-1}{N})n|C_k^n\rangle \simeq n|C_k^n\rangle, \tag{23}$$

$$\sigma_k\sigma_k^\dagger|C_k^n\rangle = (1-\frac{n}{N})(n+1)|C_k^n\rangle \simeq (n+1)|C_k^n\rangle. \tag{24}$$

The approximations in Eqs. (22) and (3) are valid in the limit $N >> n$. Equation (3) verifies again the bosonic commutation in this limit. On the contrary, if one admits both the bosonic commutation (10) and Eq. (20), by using the commutation

$$[\sigma_k, (\sigma_k^\dagger)^n] \simeq n(\sigma_k^\dagger)^{n-1}, \tag{25}$$

it can also obtain

$$\sigma_k|C_k^n\rangle \simeq \sqrt{n}|C_k^{n-1}\rangle. \tag{26}$$

The atomic storage states are also the eigenstates of the population operators

$$\sum_{j=1}^{N}(|b_j\rangle\langle b_j|C_q^n\rangle = (N-n)|C_q^n\rangle, \tag{27}$$

$$\sum_{j=1}^{N}(|c_j\rangle\langle c_j|C_q^n\rangle = n|C_q^n\rangle. \tag{28}$$

(see appendix A)

According to Dicke's definition (Eq. (47) in Ref. [13]), the total angular momentum operators of atomic ensemble can be described as

$$R_{k1} = (\sqrt{N}/2)(\sigma_k^\dagger + \sigma_k), \tag{29}$$

$$R_{k2} = (-i\sqrt{N}/2)(\sigma_k^\dagger - \sigma_k), \tag{30}$$

$$R_3 = (N/2)(\sigma_k^\dagger\sigma_k - \sigma_k\sigma_k^\dagger), \tag{31}$$

$$\begin{aligned} R^2 &= R_{k1}^2 + R_{k2}^2 + R_3^2 \\ &= (N/2)(\sigma_k^\dagger\sigma_k + \sigma_k\sigma_k^\dagger) + (N^2/4)(\sigma_k^\dagger\sigma_k - \sigma_k\sigma_k^\dagger)^2. \end{aligned} \tag{32}$$

Using the exact relations of Eqs. (18) and (22), one obtains

$$R_3|C_q^n\rangle = \frac{1}{2}(2n - N)|C_q^n\rangle, \tag{33}$$

$$R^2|C_q^n\rangle = \frac{1}{2}N(\frac{1}{2}N + 1)|C_q^n\rangle. \tag{34}$$

Therefore, the atomic storage state defined in Eq. (15) is the right Dicke state with the maximum cooperation number $r = N/2$. The discussion in this section exploits a new feature of Dicke state. The Dicke states with the maximum cooperation number play the role of number states in front of the collective lower and upper atomic operators.

4 multimode atomic storage states

For the multimode case, it concerns how the information of the multimode photons is distributed in the local atomic excitations $|c_j\rangle$. To see it, we firstly consider a simple case — the multimode single-excitation atomic storage state, that is, each mode contains only one excitation. We apply the multimode creation operators to the "vacuum" state

$$\begin{aligned} &\sigma_{k_1}^\dagger \cdots \sigma_{k_n}^\dagger |C^0\rangle \\ = &\sum_{i_1=1}^{N} |c_{i_1}\rangle\langle b_{i_1}| \exp(ik_1 z_{i_1}) \cdots \sum_{i_n=1}^{N} |c_{i_n}\rangle\langle b_{i_n}| \exp(ik_n z_{i_n}) |b_1 b_2 \cdots b_N\rangle \\ = &\frac{1}{\sqrt{N^n}} \sum_{\{i_n\}}{}' |b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle \exp[i(k_1 z_{i_1} + \cdots + k_n z_{i_n})]. \end{aligned} \tag{35}$$

This equation is apparently different of Eq. (12) by the phase factors. Indeed, the exchanges of the indices in the summation contribute to a same atomic state, but, with different phase distributions. For example, $(i_1 = 1, i_2 = 2, i_3, \cdots, i_n)$ and $(i_1 = 2, i_2 = 1, i_3, \cdots, i_n)$ display the same state $|c_1 c_2 b_3 \cdots c_{i_3} \cdots c_{i_n} \cdots b_N\rangle$, but with the phase factors $\exp[i(k_1 z_1 + k_2 z_2 + k_3 z_{i_3} + \cdots + k_n z_{i_n})]$ and $\exp[i(k_1 z_2 + k_2 z_1 + k_3 z_{i_3} + \cdots + k_n z_{i_n})]$, respectively. Mathematically, for a given atomic collective state $|b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle$, it allocates $n!$ phase factors due to $n!$ permutations for n elements. This means that an atom in the level $|c_j\rangle$, located at position z_j, records the information of all the modes. Because field is global, each atom in medium experiences the field circumstance of all the modes, and, vice versa, the field of each mode affects to all the excited atoms.

To simplify the sign, we designate

$$\{k_n\} \cdot \{z_{i_n}\}_l \equiv (k_1 z_{i_1} + \cdots + k_n z_{i_n})_l, \tag{36}$$

where $\{z_{i_n}\}_l$ stands for the l-th sequence of all the $n!$ permutations for n elements. Accordingly, Eq. (35) can be written as

$$\sigma_{k_1}^\dagger \cdots \sigma_{k_n}^\dagger |C^0\rangle = \frac{1}{\sqrt{N^n}} \sum_{\{i_n\}}{}'' |b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle \sum_{l=1}^{n!} \exp[i\{k_n\} \cdot \{z_{i_n}\}_l], \tag{37}$$

where $\sum''_{\{i_n\}}$ has been defined in Eq. (14). In comparison with the single mode case, shown in Eq. (13), n-excitation in Eq. (37) shares the phases of n modes. We define a multimode single-excitation atomic storage state as

$$\begin{aligned} |C_{k_1}^1 \cdots C_{k_n}^1\rangle \equiv & \frac{1}{\sqrt{N(N-1)\cdots(N-n+1)}} \\ & \times \sum_{\{i_n\}}{}'' |b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle \sum_{l=1}^{n!} \exp[i\{k_n\} \cdot \{z_{i_n}\}_l], \end{aligned} \tag{38}$$

which has been normalized, as shown in Appendix B. In combination of Eqs. (37) and (38), we obtain

$$\sigma_{k_1}^\dagger \cdots \sigma_{k_n}^\dagger |C^0\rangle = \sqrt{\frac{N(N-1)\cdots(N-n+1)}{N^n}} |C_{k_1}^1 \cdots C_{k_n}^1\rangle \simeq |C_{k_1}^1 \cdots C_{k_n}^1\rangle. \tag{39}$$

Now, we discuss the general case of multimode atomic storage state. For s modes containing total n excitations, it can be generated by

$$(\sigma_{k_1}^\dagger)^{m_1} \cdots (\sigma_{k_s}^\dagger)^{m_s} |C^0\rangle \tag{40}$$

$$= \frac{1}{\sqrt{N^n}}\left(\sum_{j=1}^{N}|c_j\rangle\langle b_j|\exp(ik_1z_j)\right)^{m_1}\cdots\left(\sum_{j=1}^{N}|c_j\rangle\langle b_j|\exp(ik_sz_j)\right)^{m_s}|b_1b_2\cdots b_N\rangle$$

where $m_1+\cdots+m_s=n$. By defining an index set as

$$\{i_n\}\equiv(i_1,\cdots,i_{m_1},i_{m_1+1},\cdots,i_{m_2},\cdots,i_{n-m_s+1},\cdots,i_n), \tag{41}$$

Eq. (40) can be written as

$$\begin{aligned}&(\sigma^\dagger_{k_1})^{m_1}\cdots(\sigma^\dagger_{k_s})^{m_s}|C^0\rangle \\ =\;& \frac{1}{\sqrt{N^n}}\sum_{\{i_n\}}{}'|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \\ &\times\exp[ik_1(z_{i_1}+\cdots+z_{i_{m_1}})+\cdots+ik_s(z_{i_{n-m_s+1}}+\cdots+z_{i_n})].\end{aligned} \tag{42}$$

Similarly as in the previous cases, any particular atomic collective state $|b_1\cdots c_{j_1}\cdots c_{j_n}\cdots b_N\rangle$ is related to $n!$ terms in the summation throughout all indices. But, among these $n!$ terms, the phase factor of each term will repeatedly appear $m_1!\cdots m_s!$ times because exchanges of indices within a mode cause no difference. In result, the remain non-repeated phase factors are $n!/(m_1!\cdots m_s!)$ terms. We designate again

$$\{k_s^{(m_s)}\}\cdot\{z_{i_n}\}_l\equiv\Big(k_1(z_{i_1}+\cdots+z_{i_{m_1}})+\cdots+k_s(z_{i_{n-m_s+1}}+\cdots+z_{i_n})\Big)_l \tag{43}$$

as one of these combinations with index l. Equation (42) is written as

$$\begin{aligned}&(\sigma^\dagger_{k_1})^{m_1}\cdots(\sigma^\dagger_{k_s})^{m_s}|C^0\rangle \\ =\;& \frac{m_1!\cdots m_s!}{\sqrt{N^n}}\sum_{\{i_n\}}{}''|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle\sum_{l=1}^{n!/(m_1!\cdots m_s!)}\exp[i\{k_s^{(m_s)}\}\cdot\{z_{i_n}\}_l].\end{aligned} \tag{44}$$

A general multimode atomic storage state is defined as

$$\begin{aligned}&|C_{k_1}^{m_1}\cdots C_{k_s}^{m_s}\rangle \\ \equiv\;& \sqrt{\frac{m_1!\cdots m_s!}{N(N-1)\cdots(N-n+1)}} \\ &\times\sum_{\{i_n\}}{}''|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle\sum_{l=1}^{n!/(m_1!\cdots m_s!)}\exp[i\{k_s^{(m_s)}\}\cdot\{z_{i_n}\}_l],\end{aligned} \tag{45}$$

which has been normalized (see Appendix B). Finally, Eq. (44) becomes

$$\begin{aligned} & (\sigma_{k_1}^\dagger)^{m_1} \cdots (\sigma_{k_s}^\dagger)^{m_s} |C^0\rangle \\ = & \sqrt{\frac{N(N-1)\cdots(N-n+1)}{N^n}} \sqrt{m_1! \cdots m_s!} |C_{k_1}^{m_1} \cdots C_{k_s}^{m_s}\rangle \\ \simeq & \sqrt{m_1! \cdots m_s!} |C_{k_1}^{m_1} \cdots C_{k_s}^{m_s}\rangle, \end{aligned} \tag{46}$$

where the approximation is valid in the limit $N >> n$. In this limit, the creation of an excitation of mode l for a multimode storage state is written as

$$\sigma_{k_l}^\dagger |C_{k_1}^{m_1} \cdots C_{k_l}^{m_l} \cdots C_{k_s}^{m_s}\rangle \simeq \sqrt{m_l+1} |C_{k_1}^{m_1} \cdots C_{k_l}^{m_l+1} \cdots C_{k_s}^{m_s}\rangle. \tag{47}$$

Similarly as in the single mode case, by considering the bosonic commutation (10) and Eq. (46), the annihilation of an excitation of mode l is written as

$$\sigma_{k_l} |C_{k_1}^{m_1} \cdots C_{k_l}^{m_l} \cdots C_{k_s}^{m_s}\rangle \simeq \sqrt{m_l} |C_{k_1}^{m_1} \cdots C_{k_l}^{m_l-1} \cdots C_{k_s}^{m_s}\rangle. \tag{48}$$

Equations (17), (39) and (46) imply that the state $|C^0\rangle$ defined in Eq. (11) represents a vacuum state not only for single mode but also for multimode. Physically, it shows that the ultracold collective atoms are able to store a multimode field. In correspondence, we may derive

$$\sigma_{k_2}^\dagger |C_{k_1}^{m_1}\rangle \simeq |C_{k_2} C_{k_1}^{m_1}\rangle. \tag{49}$$

$|C_{k_1}^{m_1}\rangle$ can be understood as either a single mode state or a multimode state with the vacuum for those modes other than mode k_1.

In the above theoretical description, we have shown that the atomic storage states are duplicate of Fock states of electromagnetic field. The explicit expressions of atomic storage states Eqs. (15) and (45) display duality of particle and coherence. The excitations may appear everywhere with an equal probability in medium in correspondence with the nonlocality for photons. However, each excitation takes the local phase factors of single or multimode fields as the quantum probability amplitude recording the coherence. Consequently, it may establish the correspondence of two quantum system, the field and the atomic ensemble. This provides the basis for a complete storage of quantum information of bosonic field in atomic medium.

5 dark states in EIT

In the EIT configuration, the weak signal field interacting resonantly with the atomic transition $|a\rangle - |b\rangle$ is described by the field operator

$$\begin{aligned} E_s(z,t) &= (1/2)\mathcal{E}_0 a(z)\exp[i(k_s z - \omega_s t)] + \text{h.c.} \\ &= (1/2)\mathcal{E}_0 \sum_q a(q)\exp[iqz]\exp[i(k_s z - \omega_s t)] + \text{h.c.}, \end{aligned} \tag{50}$$

where $\mathcal{E}_0$ is the field amplitude per photon and $\omega_s = ck_s$. c is the light speed in vacuum. The strong control field driving resonantly the atomic transition $|a\rangle - |c\rangle$ is assumed as classical

$$E_c(z,t) = \frac{\hbar\Omega}{2\wp_{ac}}\exp[i(k_c z - \omega_c t)] + \text{c.c.}, \tag{51}$$

where $\wp_{ac}$ is the dipole moment of the transition $|a\rangle - |c\rangle$ and $\omega_c = ck_c$. In the interaction picture, the interaction Hamiltonian is described as

$$\begin{aligned} H_I &= \hbar\sum_q \omega_q a^\dagger(q)a(q) \\ &\quad -\frac{\hbar}{2}\sum_{j=1}^{N}\left\{g\sum_q a(q)|a_j\rangle\langle b_j|\exp[i(k_s+q)z_j] + \Omega|a_j\rangle\langle c_j|\exp[ik_c z_j] + \text{h.c.}\right\}, \end{aligned} \tag{52}$$

where $\omega_q = cq$ is the detuning of mode q with respect to the resonant frequency ω_s of the signal field. By defining the atomic collective operators

$$\rho_{ab}(q) = \frac{1}{N}\sum_{j=1}^{N}|a_j\rangle\langle b_j|\exp[i(k_s+q)z_j], \tag{53}$$

$$\rho_{ac}(q) = \frac{1}{N}\sum_{j=1}^{N}|a_j\rangle\langle c_j|\exp[i(k_c+q)z_j], \tag{54}$$

the Hamiltonian (52) is written as

$$H_I = \hbar\sum_q \omega_q a^\dagger(q)a(q) - \frac{\hbar}{2}\left\{gN\sum_q a(q)\rho_{ab}(q) + \Omega N\rho_{ac}(0) + \text{h.c.}\right\}. \tag{55}$$

The new quantum field operator defined in Ref. [3] is written as

$$\psi_q = \cos\theta a_q - \sin\theta\sigma_q, \tag{56}$$

where

$$\cos\theta = \Omega/\sqrt{\Omega^2 + g^2 N}, \qquad \sin\theta = g\sqrt{N}/\sqrt{\Omega^2 + g^2 N}. \tag{57}$$

The transition $|b\rangle - |c\rangle$ concerns both the absorption of a signal photon and the emission of a driving photon. Replacing k by $(k_s + q) - k_c$ in Eq. (2), one obtains the annihilation operator σ_q of the collective atoms in EIT

$$\sigma_q = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |b_j\rangle\langle c_j| \exp[-i(k_s + q - k_c)z_j]. \tag{58}$$

In correspondence, k should be also replaced by $k_s + q - k_c$ in the atomic storage states. ψ_q satisfies the bosonian commutation relation as long σ_q does

$$[\psi_q, \psi_{q'}^\dagger] = \cos^2\theta[a_q, a_{q'}^\dagger] + \sin^2\theta[\sigma_q, \sigma_{q'}^\dagger] \simeq \delta_{qq'}. \tag{59}$$

It has been shown in Eq. (57), that the parameter θ is related to the strength of the control field. In the strong and weak limits of the control field, ψ_q tends to a_q and σ_q, respectively.

According to Ref. [3], the dark state is defined as

$$|D_q^n\rangle = \frac{1}{\sqrt{n!}}(\psi_q^\dagger)^n |0\rangle|C^0\rangle, \tag{60}$$

where $|0\rangle$ is the vacuum state of the signal field. The lowest dark state is designated as $|D^0\rangle \equiv |0\rangle|C^0\rangle$. The quasi-particle in the dark state is called polariton [3].

Using Eq. (17), one obtains the exact expression of the dark state

$$\begin{aligned} |D_q^n\rangle &= \sum_{m=0}^{n} (-1)^m \sqrt{\frac{n(n-1)\cdots(n-m+1)}{m!}} \sqrt{\frac{N(N-1)\cdots(N-m+1)}{N^m}} \\ &\quad \times \cos^{n-m}\theta \sin^m\theta |n-m\rangle|C_q^m\rangle \\ &= \sum_{m=0}^{n} \sqrt{\frac{n(n-1)\cdots(n-m+1)}{m!}} \sqrt{\frac{N(N-1)\cdots(N-m+1)}{N^m}} \\ &\quad \times \frac{\Omega^{n-m}(-g\sqrt{N})^m}{(\Omega^2 + g^2 N)^{n/2}} |n-m\rangle|C_q^m\rangle. \end{aligned} \tag{61}$$

The dark states described above are orthogonal each other since they have different quasiparticles number, but not normalized. Under the condition

$N >> n$, the dark state can be approximately written as

$$
\begin{aligned}
|D_q^n\rangle &\simeq \sum_{m=0}^{n}(-1)^m\sqrt{\frac{n(n-1)\cdots(n-m+1)}{m!}}\cos^{n-m}\theta\sin^m\theta|n-m\rangle|C_q^m\rangle \\
&= \sum_{m=0}^{n}\sqrt{\frac{n(n-1)\cdots(n-m+1)}{m!}}\frac{\Omega^{n-m}(-g\sqrt{N})^m}{(\Omega^2+g^2N)^{n/2}}|n-m\rangle|C_q^m\rangle. \quad (62)
\end{aligned}
$$

The above expression of the dark state satisfies the normalized orthogonal relation

$$\langle D_q^n|D_q^m\rangle = \delta_{nm}. \quad (63)$$

In Eq. (62), it shows that when the parameter θ is taken as 0 and $\pi/2$, the summation in the dark states reduces to only the first and the last term

$$
\begin{aligned}
|D_q^n\rangle &= |n\rangle|C^0\rangle \qquad \text{for} \quad \theta = 0, & (64)\\
|D_q^n\rangle &= (-1)^n|0\rangle|C_q^n\rangle \qquad \text{for} \quad \theta = \pi/2, & (65)
\end{aligned}
$$

respectively. Therefore, by varying θ adiabatically, the photon state and the corresponding atomic storage state can be transmitted each other.

According to definition (60), one can obtain the exact expression

$$\psi_q^\dagger|D_q^n\rangle = \frac{1}{\sqrt{n!}}(\psi_q^\dagger)^{n+1}|0\rangle|C^0\rangle = \sqrt{n+1}|D_q^{n+1}\rangle. \quad (66)$$

It is easy to check

$$\psi_q|D^0\rangle = 0. \quad (67)$$

As the same for the operator σ_q, with the help of the bosonic commutation relation (59), one obtains for the dark state (62)

$$\psi_q|D_q^n\rangle \simeq \sqrt{n}|D_q^{n-1}\rangle. \quad (68)$$

Moreover, the multimode dark state can be generated by

$$|D_{q_1}^{n_1}\cdots D_{q_s}^{n_s}\rangle = \frac{1}{\sqrt{n_1!\cdots n_s!}}(\psi_{q_1}^\dagger)^{n_1}\cdots(\psi_{q_s}^\dagger)^{n_s}|0\rangle|C^0\rangle. \quad (69)$$

They can be treated just like the multimode photon number states.

In Appendix C, we have proven that, at the exact resonance, both the exact expression (61) and the approximate expression (62) of the dark states are the eigenstates of the interaction Hamiltonian (55) with a null eigenvalue.

A pulse of monochromic light has a narrow bandwidth, and the detuning ω_q from the carrier frequency ω_s is small. If we omit the first term in Hamiltonian (55), the multimode dark states consisting the pulse are the eigenstates of the interaction Hamiltonian.

Assume that, at the initial time, a signal pulse is at a multimode state

$$\sum_{\{q_s\}} \alpha(q_1, \cdots, q_s)|n_1 \cdots n_s\rangle, \tag{70}$$

while the cold collective atoms are in the ground state $|C^0\rangle$. The combined system of the signal field and the atoms is in the state

$$|\Psi(0)\rangle = \sum_{\{q_s\}} \alpha(q_1, \cdots, q_s)|n_1 \cdots n_s\rangle|C^0\rangle = \sum_{\{q_s\}} \alpha(q_1, \cdots, q_s)|D_{q_1}^{n_1} \cdots D_{q_s}^{n_s}\rangle_{\theta=0}. \tag{71}$$

When the control field is enough strong, the signal pulse can maintain and transmit through the medium. Note that $|\Psi(0)\rangle$ is also the eigenstates of the interaction Hamiltonian with a null eigenvalue. If the control field is changed adiabatically to a very small level at a later time t_1, the state of the system is also changed adiabatically to

$$\begin{aligned} |\Psi(t_1)\rangle &= \sum_{\{q_s\}} \alpha(q_1, \cdots, q_s)|D_{q_1}^{n_1} \cdots D_{q_s}^{n_s}\rangle_{\theta=\pi/2} \qquad (72) \\ &= \sum_{\{q_s\}} (-1)^{n_1+\cdots+n_s} \alpha(q_1, \cdots, q_s)|0\rangle|C_{q_1}^{n_1} \cdots C_{q_s}^{n_s}\rangle. \end{aligned}$$

It forms an associate state for $|\Psi(0)\rangle$. The whole quantum information of the signal pulse has been stored in the atomic medium, in the form of a "negative copy", in which each excitation changes a π-phase. As soon as the control field regains to the previous level, the state (71) is recovered. Conversely, if Eq. (72) is an initial state generated in other model, by turning on the control field, it will be converted to the corresponding optical field to be seen.

6 conclusion

In conclusion, we define collective atomic operators and atomic storage states by containing spatial coherence and illustrate the conditions under which the multimode collective atomic lower and upper operators are boson-like ones. We indicate the fact that the atomic storage states shown by definition (15) are the Dicke states with the maximum cooperation number. What the new feature for these Dicke states is that, in the low excitation limit for a large number of atoms, they behave as the Fock states of electromagnetic field. The complete description and the deductive explicit expressions for the atomic storage states present better physical understanding why the atomic ensemble can record full quantum information, both excitation and coherence, for an optical field. In this connection, a complete transfer of quantum state in EIT can be established between two quantum systems: radiation field and atomic ensemble.

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appendix A

First, we derive the exact equation (22). By applying definitions (2) and (15), it gives

$$\begin{aligned}\sigma_k|C_k^n\rangle &= \frac{1}{\sqrt{N}}\sum_{l=1}^{N}|b_l\rangle\langle c_l|\exp[-ikz_l]\sqrt{\frac{n!}{N\cdots(N-n+1)}}A1 \\ &\quad\times\sum_{\{i_n\}}{}''|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle\exp[ik(z_{i_1}+\cdots+z_{i_n})].\end{aligned} \tag{73}$$

When the operator $\sum_{l=1}^{N}|b_l\rangle\langle c_l|$ applies to a particular state $|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle$, it produces a superposition of n states in which $|c_{i_k}\rangle$ is sequentially replaced by $|b_{i_k}\rangle$. After this operation, Eq. (73) is a summation of $n\times\binom{N}{n}=N(N-1)\cdots(N-n+1)/(n-1)!$ states, in which $n-1$ atoms are populated in the level $|c\rangle$. It is in fact $N-n+1$ times $|C_q^{n-1}\rangle$. For example, a particular state in $|C_q^{n-1}\rangle$, say, $|c_1\cdots c_{n-1}b_n\cdots b_N\rangle$, comes from $N-(n-1)$ states, $|c_1\cdots c_{n-1}c_nb_{n+1}\cdots b_N\rangle$, $|c_1\cdots c_{n-1}b_nc_{n+1}\cdots b_N\rangle$, ..., $|c_1\cdots c_{n-1}b_nb_{n+1}\cdots c_N\rangle$, in $|C_q^n\rangle$. Thus, Eq. (73) is written as

$$\sigma_k|C_k^n\rangle = \frac{1}{\sqrt{N}}\sqrt{\frac{n!}{N\cdots(N-n+1)}}(N-n+1)A2 \tag{74}$$

$$\times \sum_{\{i_{n-1}\}}{}'' |b_1 \cdots c_{i_1} \cdots c_{i_{n-1}} \cdots b_N\rangle \exp[ik(z_{i_1} + \cdots + z_{i_{n-1}})]$$

$$= \frac{1}{\sqrt{N}} \sqrt{\frac{n!}{N \cdots (N-n+1)}} (N-n+1) \sqrt{\frac{N \cdots (N-n+2)}{(n-1)!}} |C_q^{n-1}\rangle$$

$$= \sqrt{\frac{N-n+1}{N}} \sqrt{n} |C_q^{n-1}\rangle.$$

Then, we prove Eqs. (3). Equation (27) is written as

$$\sum_{l=1}^{N} (|b_l\rangle\langle b_l|C_q^n\rangle A3 \tag{75}$$

$$= \sqrt{\frac{n!}{N \cdots (N-n+1)}}$$

$$\times \sum_{l=1}^{N} \sum_{\{i_n\}}{}'' |b_l\rangle\langle b_l|b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle \exp[ik(z_{i_1} + \cdots + z_{i_n})]$$

$$= \sqrt{\frac{n!}{N \cdots (N-n+1)}} \sum_{l=1}^{N} \sum_{\{i_n\}}{}'' |b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle$$

$$\times \exp[ik(z_{i_1} + \cdots + z_{i_n})](1 - \delta_{li_1}) \cdots (1 - \delta_{li_n}).$$

Because all the indices of i_k are not equal each other, one has

$$(1-\delta_{li_1}) \cdots (1-\delta_{li_n}) = 1-(\delta_{li_1}+\cdots+\delta_{li_n})+(\delta_{li_1}\delta_{li_2}+\cdots)-\cdots = 1-(\delta_{li_1}+\cdots+\delta_{li_n}).A4 \tag{76}$$

Substituting Eq. (76) into Eq. (75), one obtains

$$\sum_{l=1}^{N} (|b_l\rangle\langle b_l|C_q^n\rangle = (N-n)|C_q^n\rangle.A5 \tag{77}$$

Eq. (28) is proved as

$$\sum_{l=1}^{N} (|c_l\rangle\langle c_l|C_q^n\rangle A6 \tag{78}$$

$$= \sqrt{\frac{n!}{N \cdots (N-n+1)}}$$

$$\times \sum_{l=1}^{N} \sum_{\{i_n\}}{}'' |c_l\rangle\langle c_l|b_1 \cdots c_{i_1} \cdots c_{i_n} \cdots b_N\rangle \exp[ik(z_{i_1} + \cdots + z_{i_n})]$$

$$
\begin{aligned}
&= \sqrt{\frac{n!}{N\cdots(N-n+1)}} \\
&\quad \times \sum_{l=1}^{N}\sum_{\{i_n\}}{}'' |b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \exp[ik(z_{i_1}+\cdots+z_{i_n})](\delta_{li_1}+\cdots+\delta_{li_n}) \\
&= n|C_q^m\rangle.
\end{aligned}
$$

appendix B

In this appendix, we calculate the normalized coefficient of the multimode storage states. A multimode single-excitation state is defined as

$$
|C^1_{k_1}\cdots C^1_{k_n}\rangle \equiv \alpha \sum_{\{i_n\}}{}''|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \sum_{l=1}^{n!}\exp[i\{k_n\}\cdot\{z_{i_n}\}_l].B1 \quad (79)
$$

The probability of each state in the above superposition is

$$
\begin{aligned}
\left|\sum_{l=1}^{n!}\exp[i\{k_n\}\cdot\{z_{i_n}\}_l]\right|^2 &= \sum_{l,j}^{n!}\exp[i\{k_n\}\cdot(\{z_{i_n}\}_l-\{z_{i_n}\}_j)]B2 \quad (80) \\
&= n!+\sum_{l\neq j}^{n!}\exp[i\{k_n\}\cdot(\{z_{i_n}\}_l-\{z_{i_n}\}_j)].
\end{aligned}
$$

Then, we sum to all these probabilities. For the first term of Eq. (80), it is simply as

$$
\sum_{\{i_n\}}{}''n! = n!\frac{N(N-1)\cdots(N-n+1)}{n!} = N(N-1)\cdots(N-n+1).B3 \quad (81)
$$

In the multimode case, one must find a mode with $k_j \neq 0$. By using Eq. (9), the summation $\sum''_{\{i_n\}}$ to the second term of Eq. (80) vanishes. Therefore, we obtain

$$
\alpha = \frac{1}{\sqrt{N(N-1)\cdots(N-n+1)}}.B4 \quad (82)
$$

Similarly, for a general multimode storage state defined by Eq. (45), the probability finding a single state is

$$
\left|\sum_{l=1}^{n!/(m_1!\cdots m_s!)}\exp[i\{k_s^{(m_s)}\}\cdot\{z_{i_n}\}_l]\right|^2 B5 \quad (83)
$$

$$
\begin{aligned}
&= \sum_{l,j}^{n!/(m_1!\cdots m_s!)} \exp[i\{k_s^{(m_s)}\}\cdot(\{z_{i_n}\}_l - \{z_{i_n}\}_j)] \\
&= \frac{n!}{m_1!\cdots m_s!} + \sum_{l\neq j}^{n!/(m_1!\cdots m_s!)} \exp[i\{k_s^{(m_s)}\}\cdot(\{z_{i_n}\}_l - \{z_{i_n}\}_j)].
\end{aligned}
$$

The summation to the first term of the above equation gives

$$
\begin{aligned}
&\sum_{\{i_n\}}{}'' \frac{n!}{m_1!\cdots m_s!} B6 \qquad (84)\\
= \;& \frac{n!}{m_1!\cdots m_s!}\frac{N(N-1)\cdots(N-n+1)}{n!} = \frac{N(N-1)\cdots(N-n+1)}{m_1!\cdots m_s!}.
\end{aligned}
$$

With the same reason, the summation to the second term vanishes. The normalized coefficient is therefore

$$
\alpha = \sqrt{\frac{m_1!\cdots m_s!}{N(N-1)\cdots(N-n+1)}}.B7 \qquad (85)
$$

appendix C

We define a new collective atomic state in which n atoms are in the level $|c\rangle$ whereas one atom is in the level $|a\rangle$

$$
\begin{aligned}
|A_q^1, C_q^n\rangle \;\equiv\; & \sqrt{\frac{n!}{N(N-1)\cdots(N-n)}} \sum_{l\neq\{i_n\}} \sum_{\{i_n\}}{}'' |b_1\cdots a_l\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \\
& \times \exp[i(k_s+q-k_c)(z_{i_1}+\cdots+z_{i_n})]\exp[i(k_s+q)z_l], C1 \qquad (86)
\end{aligned}
$$

where $\sum_{l\neq\{i_n\}}$ designates the summation for index l which cannot be taken as $i_1,\cdots i_n$. We have already indicated that, the state $|C_q^n\rangle$ is a superposition of $N(N-1)\cdots(N-n+1)/n!$ possible states $|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle$ in which n atoms are in the level $|c\rangle$ whereas the remaining $N-n$ atoms in the level $|b\rangle$. For one of these states, each of $N-n$ atoms being in the level $|b\rangle$ can be excited to the level $|a\rangle$. So the state $|A_q^1, C_q^n\rangle$ includes $N(N-1)\cdots(N-n)/n!$ such possible states $|b_1\cdots a_l\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle$ with the equal possibility. The state $|A_q^1, C_q^n\rangle$ has been normalized. The phase factor related to the excited atom l being in level $|a\rangle$ is $\exp[i(k_s+q)z_l]$, because the transition of the level $|a\rangle$ to the ground level $|b\rangle$ is connected with the signal field of the wavevector $k_s + q$. State (86) can be obtained by the following operation

$$
\begin{aligned}
& N\rho_{ac}(0)|C_q^n\rangle \\
= & \left(\sum_{l=1}^{N} |a_l\rangle\langle c_l| \exp[ik_c z_l]\right) |C_q^n\rangle C2 \qquad (87) \\
= & \sqrt{\frac{n!}{N\cdots(N-n+1)}} \sum_{l=1}^{N} \sum_{\{i_n\}}{}'' |a_l\rangle\langle c_l|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \\
& \times \exp[i(k_s+q-k_c)(z_{i_1}+\cdots+z_{i_n})] \exp[ik_c z_l] \\
= & \sqrt{\frac{n!}{N\cdots(N-n+1)}} \sum_{l\neq\{i_{n-1}\}} \sum_{\{i_{n-1}\}}{}'' |b_1\cdots a_l\cdots c_{i_1}\cdots c_{i_{n-1}}\cdots b_N\rangle \\
& \times \exp[i(k_s+q-k_c)(z_{i_1}+\cdots+z_{i_{n-1}})] \exp[i(k_s+q)z_l] \\
= & \sqrt{\frac{n!}{N\cdots(N-n+1)}} \sqrt{\frac{N\cdots(N-n+1)}{(n-1)!}} |A_q^1, C_q^{n-1}\rangle \\
= & \sqrt{n}|A_q^1, C_q^{n-1}\rangle,
\end{aligned}
$$

where the atomic operator $\rho_{ac}(0)$ has been defined in Eq. (54). Similarly, we have

$$
\begin{aligned}
& N\rho_{ab}(q)|C_q^n\rangle \\
= & \left(\sum_{l=1}^{N} |a_l\rangle\langle b_l| \exp[i(k_s+q)z_l]\right) |C_q^n\rangle C3 \qquad (88) \\
= & \sqrt{\frac{n!}{N\cdots(N-n+1)}} \sum_{l=1}^{N} \sum_{\{i_n\}}{}'' |a_l\rangle\langle b_l|b_1\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \\
& \times \exp[i(k_s+q-k_c)(z_{i_1}+\cdots+z_{i_n})] \exp[i(k_s+q)z_l] \\
= & \sqrt{\frac{n!}{N\cdots(N-n+1)}} \sum_{l\neq\{i_n\}} \sum_{\{i_n\}}{}'' |b_1\cdots a_l\cdots c_{i_1}\cdots c_{i_n}\cdots b_N\rangle \\
& \times \exp[i(k_s+q-k_c)(z_{i_1}+\cdots+z_{i_n})] \exp[i(k_s+q)z_l] \\
= & \sqrt{\frac{n!}{N\cdots(N-n+1)}} \sqrt{\frac{N\cdots(N-n)}{n!}} |A_q^1, C_q^n\rangle \\
= & \sqrt{N-n}|A_q^1, C_q^n\rangle.
\end{aligned}
$$

The two interactions induced by two fields in the interaction Hamiltonian interfere destructively for the dark state. Using Eqs. (88) and (89), for the exact expression of the dark state Eq. (61), one obtains

$$
\begin{aligned}
& \Omega N\rho_{ac}(0)|D_q^n\rangle C4 \\
= \;& \Omega N\rho_{ac}(0)\sum_{m=0}^{n}(-1)^m\sqrt{\frac{n(n-1)\cdots(n-m+1)}{m!}} \\
& \times\sqrt{\frac{N(N-1)\cdots(N-m+1)}{N^m}}\frac{\Omega^{n-m}(g\sqrt{N})^m}{(\Omega^2+g^2N)^{n/2}}|n-m\rangle|C_q^m\rangle \\
= \;& \sum_{m=1}^{n}(-1)^m\sqrt{\frac{n(n-1)\cdots(n-m+1)}{(m-1)!}}\sqrt{\frac{N(N-1)\cdots(N-m+1)}{N^m}} \\
& \times\frac{\Omega^{n-m+1}(g\sqrt{N})^m}{(\Omega^2+g^2N)^{n/2}}|n-m\rangle|A_q^1,C_q^{m-1}\rangle.
\end{aligned}
\tag{89}
$$

and

$$
\begin{aligned}
& gNa(q)\rho_{ab}(q)|D_q^n\rangle C5 \\
= \;& gNa(q)\rho_{ab}(q)\sum_{m=0}^{n}(-1)^m\sqrt{\frac{n(n-1)\cdots(n-m+1)}{m!}} \\
& \times\sqrt{\frac{N(N-1)\cdots(N-m+1)}{N^m}}\frac{\Omega^{n-m}(g\sqrt{N})^m}{(\Omega^2+g^2N)^{n/2}}|n-m\rangle|C_q^m\rangle \\
= \;& \sum_{m=0}^{n-1}(-1)^m\sqrt{\frac{n(n-1)\cdots(n-m)}{m!}}\sqrt{\frac{N(N-1)\cdots(N-m)}{N^{m+1}}} \\
& \times\frac{\Omega^{n-m}(g\sqrt{N})^{m+1}}{(\Omega^2+g^2N)^{n/2}}|n-m-1\rangle|A_q^1,C_q^{m}\rangle,
\end{aligned}
\tag{90}
$$

If we set index $m\to m+1$ in Eq. (89), it is the exact same as Eq. (90) but with an opposite sign. Therefore, one obtains

$$[gNa(q)\rho_{ab}(q)+\Omega N\rho_{ac}(0)]|D_q^n\rangle = 0.C6 \tag{91}$$

In result, for the exact resonant mode $q=0$, the dark states $|D_{q=0}^n\rangle$ are the eigenstates with the null eigenvalue of the interaction Hamiltonian (55). We note that Eqs. (89) – (91) hold exactly for the exact expression of the dark state (61). For the approximate expression of the dark state (62), Eqs. (89) – (91) are also satisfied as long $N >> n$.

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